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RESTRICTED AVERAGING OPERATORS IN THE FINITE FIELD SETTING

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Abstract. In this paper we study the mapping properties of the finite field restricted averaging operators to various algebraic varieties. We derive necessary conditions for the boundedness of the generalized restricted averaging operator related to arbitrary algebraic varieties. It is shown that the necessary conditions are in fact sufficient in the specific case when the Fourier transform on varieties has enough decay estimates. Our work extends the known optimal result on regular varieties such as paraboloids and spheres to certain lower dimensional varieties.

1. Introduction

The restriction problem is one of central open problems in harmonic analysis. Much attention has been given to the problem, in part because it is closely related to other important harmonic analysis problems such as Kakeya problems and Bochner-Riesz problems. Let $d\sigma$ denote a surface measure supported on a hypersurface $H \subset \mathbb{R}^d$. The restriction problem for $H \subset \mathbb{R}^d$ is to determine exponents $1 \leq p, r \leq \infty$ such that the following restriction inequality holds:

$$\|\widehat{f}\|_{L^r(H, d\sigma)} \leq C\|f\|_{L^p(\mathbb{R}^d)}$$

where the constant $C > 0$ is independent of the test functions $f : \mathbb{R}^d \to \mathbb{C}$ and $\widehat{f}$ is the Fourier transform of the function f which is defined by

$$\widehat{f}(\xi) = \int e^{-2\pi i x\cdot\xi} f(x)\, dx.$$

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In 1967, this problem was initially posed by E.M. Stein ([18]) and it has been extensively studied (for example, see [5, 21, 1, 20, 2, 19]). In particular, Guth ([6, 7]) has recently obtained new results on the restriction problems. Notable point is that the celebrated Guth's work is based on the polynomial method which Dvir ([4]) used to give a simple proof of the finite field Kakeya conjecture. This illustrates that the study of analysis problems in finite fields may help us find a new idea to attack challenging harmonic analysis problems in the Euclidean setting.

Over the last decades, harmonic analysis problems in finite fields have been extensively developed by many researchers. In [17], Mockenhaupt and Tao initially posed the finite field restriction problem. Their results have been recently improved by some other people (for example, see [8, 12, 16, 15, 11]). Averaging problem in finite fields was introduced by Carbery, Stones, and Wright ([3]) and some sharp results on the problems were obtained in [13, 10]. In the paper [9], the first listed author proposed the restricted averaging problem which can be considered as a hybrid of the restriction problem and the averaging problem in finite fields. In this paper, we shall develop the restricted averaging problems to various algebraic varieties in the general setting. We begin by introducing the problem. Let $\mathbb{F}_q^d$ be a d-dimensional vector space over a finite field $\mathbb{F}_q$ with q elements. We always assume that the characteristic of $\mathbb{F}_q$ is sufficiently large. Given an algebraic variety $S \subset \mathbb{F}_q^d$, we shall denote by $d\sigma_s$ the normalized surface measure supported on the variety S. Namely, the mass of one point in S is given by $\frac{1}{|S|}$, where $|S|$ denotes the cardinality of the set S. Therefore, the total mass of S is one and we see that if $f : \mathbb{F}_q^d \to \mathbb{C}$, then

$$\int_S f(x)\ d\sigma_s(x) = \frac{1}{|S|}\sum_{x\in S} f(x).$$

In particular, when $S = \mathbb{F}_q^d$, we shall write dx for $d\sigma_s$ and we call dx the normalized counting measure on $\mathbb{F}_q^d$. Now, we consider another algebraic variety $V \subset \mathbb{F}_q^d$ which is also endowed with the normalized surface measure $d\sigma_v$. In this setting, we are interested in determining $1 \leq p, r \leq \infty$ such that the following inequality holds:

$$\|f * d\sigma_s\|_{L^r(V,d\sigma_v)} \leq C\|f\|_{L^p(\mathbb{F}_q^d,dx)}, \tag{1.1}$$

where the constant $C > 0$ is independent of both the functions f on $\mathbb{F}_q^d$ and q which is the cardinality of the underlying finite field $\mathbb{F}_q$. We shall name this problem as the generalized restricted averaging problem related to S and V. When $V = \mathbb{F}_q^d$, this problem is known as the averaging problem over the variety S which was initially studied in the finite field setting by Carbery, Stones, and Wright [3]. Recently, the authors in [14] studied the restricted averaging problem in the specific case when $S = V$ and $|S| \sim q^{d-1}$. They obtained optimal results on the problems in the case when S is a regular variety such as the paraboloid, the sphere in all dimensions $d \geq 2$, and the cone in odd dimensions $d \geq 3$. In addition, when S is the cone in even dimensions $d \geq 4$, the sharp results on the problem were obtained except for two endpoints. In this paper, we will deduce partial results on the problem in general varieties V and S. As a direct consequence from our results, we provide certain conditions on the varieties V and S where optimal results on the restricted averaging problem can be obtained. Our work extends the results for hyperplanes to some other lower dimensional varieties in finite fields.

2. Statement of main results

Given algebraic varieties $S, V \subset \mathbb{F}_q^d$, we define an operator $A_{S,V}$ acting on functions $f : \mathbb{F}_q^d \to \mathbb{C}$ as the restriction of the $f * d\sigma_s$ to the variety V. In other words, we see that for $x \in V$,

$$A_{S,V} f(x) = f * d\sigma_s(x) = \int_S f(x-y) d\sigma_s(y) = \frac{1}{|S|} \sum_{y \in S} f(x-y).$$

We call the operator $A_{S,V}$ the restricted averaging operator. Recall that for positive numbers A and B depending on the size of the underlying finite field $\mathbb{F}_q$, we shall write $A \lesssim B$ if there exists a constant $C > 0$ independent of $q = |\mathbb{F}_q|$ such that $A \leq CB$. We also write $A \gtrsim B$ for $B \lesssim A$. In addition, $A \sim B$ means that $A \lesssim B$ and $A \gtrsim B$. We use the notation $A_{S,V}(p \to r) \lesssim 1$ to indicate that the inequality (1.1) holds. With this notation, the restricted averaging problem for S, V is to determine $1 \leq p, r \leq \infty$ such that $A_{S,V}(p \to r) \lesssim 1$. If $d\sigma_s$ denotes the normalized surface measure on an algebraic variety $S \subset \mathbb{F}_q^d$, then the inverse Fourier transform of $d\sigma_s$, denoted by $(d\sigma_v)^\vee$, is defined by

$$(d\sigma_s)^\vee(m) = \int_S \chi(m \cdot x) \; d\sigma_v(x) = \frac{1}{|S|} \sum_{x \in S} \chi(m \cdot x) \quad \text{for } m \in \mathbb{F}_q^d,$$

where χ denotes a nontrivial additive character of $\mathbb{F}_q$. In the previous work in [14], the authors studied the restricted averaging problem to varieties $V, S \subset \mathbb{F}_q^d$ such that $S = V$ and S is a regular variety. Here, we recall that an algebraic variety $S \subset \mathbb{F}_q^d$ is called a regular variety if $|S| \sim q^{d-1}$ and $|(d\sigma_s)^\vee(m)| \lesssim q^{-\frac{(d-1)}{2}}$ for all $m \in \mathbb{F}_q^d \setminus \{(0, \ldots, 0)\}$. The authors in [14] proved the following optimal results on the restricted averaging problem for regular varieties.

THEOREM 2.1 ([14], Theorem 1.4). *Let $d\sigma_s$ be the normalized surface measure on a regular variety $S \subset \mathbb{F}_q^d$. Then we have $A_{S,S}(p \to r) \lesssim 1$ if and only if $(1/p, 1/r)$ lies on the convex hull of points $(0,0), (0,1)$, $((d-1)/d, 1)$, and $((d-1)/d, 1/d)$.*

Since the size of a regular variety $S \subset \mathbb{F}_q^d$ is similar to q^{d-1}, Theorem 2.1 is only applied to $(d-1)$-dimensional varieties. In this paper, we shall extend Theorem 2.1 to some lower dimensional cases. The following is our main result in the general setting.

THEOREM 2.2. *Let $S, V \subset \mathbb{F}_q^d$ be algebraic varieties with $|S| \sim q^s$ and $|V| \sim q^v$ for $0 \le s, v \le d-1$. Assume that there exists a positive constant β such that $|(d\sigma_s)^\vee(m)| \lesssim q^{-\frac{\beta}{2}}$ for all $m \in \mathbb{F}_q^d \setminus \{(0,\dots,0)\}$. Then if $d-\beta-v<0$, we have*

$$A_{S,V}\left(\frac{d-2s+\beta+v}{\beta+v-s} \to \frac{d-2s+\beta+v}{d-s}\right) \lesssim 1.$$

In particular, if $|S| \sim |V|$, then the theorem below follows immediately from Theorem 2.2.

THEOREM 2.3. *Let $S, V \subset \mathbb{F}_q^d$ be algebraic varieties with $|S| \sim |V| \sim q^s$ for $0 \le s \le d-1$. In addition, assume that there exists a positive constant β such that $|(d\sigma_s)^\vee(m)| \lesssim q^{-\frac{\beta}{2}}$ for all $m \in \mathbb{F}_q^d \setminus \{(0,\dots,0)\}$. Then if $d-\beta-s<0$, we have*

$$A_{S,V}\left(\frac{d-s+\beta}{\beta} \to \frac{d-s+\beta}{d-s}\right) \lesssim 1.$$

If we take $s=\beta$ in Theorem 2.3, then the following result is immediately obtained.

COROLLARY 2.4. *Let $S, V \subset \mathbb{F}_q^d$ be algebraic varieties with $|S| \sim |V| \sim q^s$ for $\frac{d}{2} < s \le d-1$. If $|(d\sigma_s)^\vee(m)| \lesssim q^{-\frac{s}{2}}$ for all $m \in \mathbb{F}_q^d \setminus \{(0,\dots,0)\}$, then*

$$A_{S,V}\left(\frac{d}{s} \to \frac{d}{d-s}\right) \lesssim 1.$$

We shall demonstrate that Corollary 2.4 essentially yields a generalized result of Theorem 2.1. More precisely, we shall prove the following.

THEOREM 2.5. *Let $S, V \subset \mathbb{F}_q^d$ be algebraic varieties with $|S| \sim |V| \sim |S \cap V| \sim q^s$ for some $\frac{d}{2} < s \leq d-1$. In addition, assume that $|(d\sigma_s)^\vee(m)| \lesssim q^{-\frac{s}{2}}$ for all $m \in \mathbb{F}_q^d \setminus \{(0,\ldots,0)\}$. Then we have $A_{S,V}(p \to r) \lesssim 1$ if and only if $(1/p,\ 1/r)$ is contained in the convex hull of points $(0,0), (0,1), (s/d,\ 1)$, and $(s/d,\ (d-s)/d)$.*

Note that if we take $s = d-1$ in Theorem 2.5, then we are able to obtain Theorem 2.1.

The rest part of this paper will be organized to prove Theorem 2.2 and Theorem 2.5. To this end, in Section 3, we deduce the necessary conditions for $A_{S,V}(p \to r) \lesssim 1$. In Section 4, we shall give the complete proofs of both Theorem 2.2 and Theorem 2.5.

3. Necessary conditions for $A_{S,V}(p \to r) \lesssim 1$

Let us denote by $A^*_{S,V}$ the adjoint operator of $A_{S,V}$. By duality, the restricted averaging inequality (1.1) is same as the following inequality

$$\|A^*_{S,V} g\|_{L^{p'}(\mathbb{F}_q^d, dx)} \leq C \|g\|_{L^{r'}(V, d\sigma_v)}, \tag{3.1}$$

where p' and r' denote the usual Hölder conjugates of p and r, respectively (for example, $1/p + 1/p' = 1$). Throughout this paper, we write $E(x)$ for the indicator function $1_E(x)$ on the set $E \subset \mathbb{F}_q^d$. To deduce the necessary conditions for the boundedness of restricted averaging operators in the general cases, we need the following lemma which gives us the explicit form of the adjoint operator $A^*_{S,V}$.

LEMMA 3.1. *Let $S, V \subset \mathbb{F}_q^d$. Then the adjoint operator $A^*_{S,V}$ of $A_{S,V}$ is given by*

$$A^*_{S,V} g(y) = \frac{q^d}{|S||V|} \sum_{x \in V} S(x-y)\ g(x),$$

where $g : V \to \mathbb{C}$ and $y \in \mathbb{F}_q^d$.

Proof. It follows from the L^2 orthogonality that

$$< A_{S,V} f,\ g >_{L^2(V, d\sigma_v)} = < f,\ A^*_{S,V} g >_{L^2(\mathbb{F}_q^d, dx)} .$$

By the definition of the inner product, we have

$$\int_V A_{S,V} f(x)\ \overline{g(x)}\ d\sigma_v(x) = \int_{\mathbb{F}_q^d} f(y)\ \overline{A^*_{S,V} g(y)}\ dy,$$

which is same as

$$\frac{1}{|V|}\sum_{x\in V} A_{S,V}f(x)\ \overline{g(x)} = \frac{1}{q^d}\sum_{y\in \mathbb{F}_q^d} f(y)\ \overline{A^*_{S,V}g(y)}.$$

Since

$$\begin{aligned} A_{S,V}f(x) &= \frac{1}{|S|}\sum_{y\in S} f(x-y) = \frac{1}{|S|}\sum_{y\in \mathbb{F}_q^d} f(x-y)S(y)\\ &= \frac{1}{|S|}\sum_{y\in \mathbb{F}_q^d} f(y)S(x-y), \end{aligned}$$

we see that

$$\sum_{y\in \mathbb{F}_q^d} f(y)\left(\frac{1}{|V||S|}\sum_{x\in V}\overline{g(x)}\ S(x-y)\right) = \frac{1}{q^d}\sum_{y\in \mathbb{F}_q^d} f(y)\ \overline{A^*_{S,V}g(y)}.$$

This implies that

$$A^*_{S,V}g(y) = \frac{q^d}{|S||V|}\sum_{x\in V} S(x-y)\ g(x).$$

The proof is now completed. $\square$

The following lemma indicates the necessary conditions for the bound $A_{S,V}(p\to r)\lesssim 1$.

Lemma 3.2. *Let $S, V \subset \mathbb{F}_q^d$ be algebraic varieties. Let $d\sigma_s$ and $d\sigma_v$ denote the normalized surface measures on S and V, respectively. Assume that $A_{S,V}(p\to r)\lesssim 1$ for $1\le p, r\le \infty$. Then we have*

$$\frac{|V\cap S|^{\frac{1}{r}}}{|V|^{\frac{1}{r}}|S|}\lesssim q^{-\frac{d}{p}} \quad \text{and} \quad q^{\frac{d}{p}}\lesssim |S|^{\frac{1}{p}}|V|^{\frac{1}{r}}. \tag{3.2}$$

In addition, if we assume that $S\cap V$ contains an affine subspace $H\subset S\cap V\subset \mathbb{F}_q^d$, then

$$\frac{q^{\frac{d}{p}}|H|^{1-\frac{1}{p}+\frac{1}{r}}}{|S||V|^{\frac{1}{r}}}\lesssim 1. \tag{3.3}$$

Proof. Since $A_{S,V}(p\to r)\lesssim 1$, it follows that

$$\|f * d\sigma_s\|_{L^r(V, d\sigma_v)} \le C\|f\|_{L^p(\mathbb{F}_q^d, dx)}. \tag{3.4}$$

By duality, we see that

$$\|A^*_{S,V}g\|_{L^{p'}(\mathbb{F}_q^d, dx)} \le C\|g\|_{L^{r'}(V, d\sigma_v)}. \tag{3.5}$$

For each $w \in \mathbb{F}_q^d$, define $\delta_w(x) = 1$ for $x = w$ and 0 otherwise. Taking $f = \delta_{\mathbf{0}}$, we see that

$$\|f\|_{L^p(\mathbb{F}_q^d,dx)} = \left(\frac{1}{q^d}\sum_{x\in\mathbb{F}_q^d}|\delta_{\mathbf{0}}(x)|^p\right)^{\frac{1}{p}} = q^{-\frac{d}{p}},$$

and

$$\begin{aligned}\|f * d\sigma_s\|_{L^r(V,d\sigma_v)} &= \left(\frac{1}{|V|}\sum_{y\in V}\left|\frac{1}{|S|}\sum_{x\in S}\delta_{\mathbf{0}}(y-x)\right|^r\right)^{\frac{1}{r}}\\ &= \left(\frac{1}{|V|}\sum_{y\in V\cap S}\frac{1}{|S|^r}\right)^{\frac{1}{r}} = \frac{|V\cap S|^{\frac{1}{r}}}{|V|^{\frac{1}{r}}|S|}.\end{aligned}$$

Hence, it follows from (3.4) that a necessary condition for the bound $A_{S,V}(p \to r) \lesssim 1$ is given by

$$\frac{|V\cap S|^{\frac{1}{r}}}{|V|^{\frac{1}{r}}|S|} \lesssim q^{-\frac{d}{p}}. \tag{3.6}$$

To find another necessary condition for $A_{S,V}(p \to r) \lesssim 1$, we shall test the inequality (3.5) by taking $g = \delta_{\mathbf{a}}$ for some $\mathbf{a} \in V$. Indeed, if $g = \delta_{\mathbf{a}}$ for some $\mathbf{a} \in V$, then we have

$$\|g\|_{L^{r'}(V,d\sigma_v)} = \left(\frac{1}{|V|}\sum_{y\in V}|\delta_{\mathbf{a}}(y)|^{r'}\right)^{\frac{1}{r'}} = |V|^{-\frac{1}{r'}} = |V|^{\frac{1}{r}-1}.$$

On the other hand, we observe that for $y \in \mathbb{F}_q^d$,

$$A^*_{S,V}g(y) = \frac{q^d}{|S||V|}\sum_{x\in V}S(x-y)\delta_{\mathbf{a}}(x) = \frac{q^d}{|S||V|}S(\mathbf{a}-y).$$

It follows from the previous observation that

$$\begin{aligned}\|A^*_{S,V}g\|_{L^{p'}(\mathbb{F}_q^d,dy)} &= \left(\frac{1}{q^d}\sum_{y\in\mathbb{F}_q^d}\left(\frac{q^d}{|S||V|}S(\mathbf{a}-y)\right)^{p'}\right)^{\frac{1}{p'}}\\ &= \left(\frac{1}{q^d}\sum_{y\in\mathbb{F}_q^d}\left(\frac{q^d}{|S||V|}S(y)\right)^{p'}\right)^{\frac{1}{p'}}\\ &= \left(\frac{|S|}{q^d}\left(\frac{q^d}{|S||V|}\right)^{p'}\right)^{\frac{1}{p'}} = q^{-\frac{d}{p'}}|S|^{\frac{1}{p'}}\frac{q^d}{|S||V|} = \frac{q^{\frac{d}{p}}}{|S|^{\frac{1}{p}}|V|}.\end{aligned}$$

Thus, taking $f=\delta_{\mathbf{a}}$ in the inequality (3.5), we have

$$\frac{q^{\frac{d}{p}}}{|S|^{\frac{1}{p}}|V|}\lesssim |V|^{\frac{1}{r}-1}.$$

Namely, another necessary condition for $A_{S,V}(p\to r)\lesssim 1$ is obtained as follows:

$$q^{\frac{d}{p}}\lesssim |S|^{\frac{1}{p}}|V|^{\frac{1}{r}}.$$

Thus, we complete the proof of the first part (3.2) of Lemma 3.2.
To prove the inequality (3.3) in Lemma 3.2, let us assume that $S\cap V$ contains an affine subspace $H\subset S\cap V\subset\mathbb{F}_q^d$. We shall test the inequality (3.5) with $g(x)=H(x)$. It follows that

$$\|g\|_{L^{r'}(V,d\sigma_v)} = \|H\|_{L^{r'}(V,d\sigma_v)} = \left(\frac{|H|}{|V|}\right)^{\frac{1}{r'}} = \left(\frac{|H|}{|V|}\right)^{\frac{r-1}{r}}. \tag{3.7}$$

Let us estimate $\|A^*_{S,V}g\|_{L^{p'}(\mathbb{F}_q^d,dx)}$ for $g(x)=H(x)$. Since $H\subset V$, we see

$$A^*_{S,V}g(x) = A^*_{S,V}H(x) = \frac{q^d}{|S||V|}\sum_{y\in H}S(y-x).$$

It follows that

$$\|A^*_{S,V}g\|_{L^{p'}(\mathbb{F}_q^d,dx)} = \|A^*_{S,V}H\|_{L^{p'}(\mathbb{F}_q^d,dx)}$$
$$= \left(\frac{1}{q^d}\sum_{x\in\mathbb{F}_q^d}|A^*_{S,V}H(x)|^{p'}\right)^{\frac{1}{p'}}$$
$$= \frac{1}{q^{\frac{d}{p'}}}\frac{q^d}{|S||V|}\left(\sum_{x\in\mathbb{F}_q^d}\left|\sum_{y\in H}S(y-x)\right|^{p'}\right)^{\frac{1}{p'}}. \tag{3.8}$$

Now, we claim that

$$\sum_{x\in\mathbb{F}_q^d}\left|\sum_{y\in H}S(y-x)\right|^{p'} \geq |H||H|^{p'}. \tag{3.9}$$

To prove this claim, let us assume that $H+\alpha := \{h+\alpha\in\mathbb{F}_q^d : h\in H\}$ is a subspace for some $\alpha\in\mathbb{F}_q^d,$. Then it is clear that $(H+\alpha)+z = H+\alpha$ for all $z\in H+\alpha$. Since $H\subset S$, we see that

$$\sum_{x\in\mathbb{F}_q^d}\left|\sum_{y\in H}S(y-x)\right|^{p'} \geq \sum_{x\in\mathbb{F}_q^d}\left|\sum_{y\in H}H(y-x)\right|^{p'}$$
$$= \sum_{x\in\mathbb{F}_q^d}\left|\sum_{y+\alpha\in H+\alpha}(H+\alpha)(y+\alpha-x)\right|^{p'}$$
$$\geq \sum_{-x\in H+\alpha}\left|\sum_{y+\alpha\in H+\alpha}(H+\alpha)(y+\alpha-x)\right|^{p'}$$
$$= \sum_{z\in H+\alpha}\left|\sum_{w\in H+\alpha}(H+\alpha)(w+z)\right|^{p'}$$
$$= |H+\alpha||H+\alpha|^{p'} = |H||H|^{p'}.$$

Combining (3.8) with (3.9), we obtain that

$$\|A^*_{S,V}g\|_{L^{p'}(\mathbb{F}_q^d,dx)} = \|A^*_{S,V}H\|_{L^{p'}(\mathbb{F}_q^d,dx)}$$
$$\geq \frac{q^{\frac{d}{p}}}{|S||V|}\left(|H||H|^{p'}\right)^{\frac{1}{p'}} = \frac{q^{\frac{d}{p}}}{|S||V|}|H|^{\frac{2p-1}{p}}.$$

Using this estimate and (3.7), we see that the inequality (3.5) yields

$$\frac{q^{\frac{d}{p}}}{|S||V|}|H|^{\frac{2p-1}{p}} \lesssim \left(\frac{|H|}{|V|}\right)^{\frac{r-1}{r}}.$$

Simplifying this inequality, we obtain the inequality (3.3) in Lemma 3.2, which gives us further necessary condition for $A_{S,V}(p \to r) \lesssim 1$ in the specific case when $S \cap V$ contains an affine subspace H. $\square$

The following corollary is direct consequences from Lemma 3.2.

Corollary 3.3. *Let $S, V \subset \mathbb{F}_q^d$ be the algebraic varieties given as in Lemma 3.2. Assume that $|S| \sim q^s$, $|V| \sim q^v$, and $|S \cap V| \sim q^c$ for $0 \le s, v, c \le d-1$. Then the necessary conditions for $A_{S,V}(p \to r) \lesssim 1$ are given by*

$$\frac{d}{p} - s \le \frac{v-c}{r} \quad \text{and} \quad \frac{d-s}{p} \le \frac{v}{r}. \tag{3.10}$$

In addition, if $S \cap V$ contains an affine subspace $H \subset \mathbb{F}_q^d$ with $|H| = q^h$, then the further necessary condition for $A_{S,V}(p \to r) \lesssim 1$ is given by

$$\frac{d-h}{p} + h - s \le \frac{v-h}{r}.$$

4. Proofs of the main theorems (Theorem 2.2 and Theorem 2.5)

In this section, we restate our main theorems introduced in Section 2 and we complete proofs of them. First, we prove Theorem 2.2.

Theorem 2.2. *Let $S, V \subset \mathbb{F}_q^d$ be algebraic varieties with $|S| \sim q^s$ and $|V| \sim q^v$ for $0 \le s, v \le d-1$. Assume that there exists a positive constant β such that $|(d\sigma_s)^\vee(m)| \lesssim q^{-\frac{\beta}{2}}$ for all $m \in \mathbb{F}_q^d \setminus \{(0,\dots,0)\}$. Then if $d - \beta - v < 0$, then we have*

$$A_{S,V}\left(\frac{d-2s+\beta+v}{\beta+v-s} \to \frac{d-2s+\beta+v}{d-s}\right) \lesssim 1.$$

Proof. Put $p_0 = \frac{d-2s+\beta+v}{\beta+v-s}$ and $r_0 = \frac{d-2s+\beta+v}{d-s}$. Then we aim to prove that the following inequality holds:

$$\|f * d\sigma_s\|_{L^{r_0}(V, d\sigma_v)} \lesssim \|f\|_{L^{p_0}(\mathbb{F}_q^d, dx)} \quad \text{for all } f : \mathbb{F}_q^d \to \mathbb{C}. \tag{4.1}$$

Define $K(m) = (d\sigma_s)^\vee(m) - \delta_{\mathbf{0}}(m)$ for $m \in \mathbb{F}_q^d$. Since $\widehat{\delta_{\mathbf{0}}}(x) = 1$ for all $x \in \mathbb{F}_q^d$, we can write

$$f * d\sigma_s = f * (\widehat{K} + \widehat{\delta_{\mathbf{0}}}) = f * \widehat{K} + f * 1.$$

To prove the inequality (4.1), it suffices to prove that the following two inequalities hold:

$$\|f * 1\|_{L^{r_0}(V,d\sigma_v)} \lesssim \|f\|_{L^{p_0}(\mathbb{F}_q^d,dx)} \quad \text{for all } f : \mathbb{F}_q^d \to \mathbb{C} \tag{4.2}$$

and

$$\|f * \widehat{K}\|_{L^{r_0}(V,d\sigma_v)} \lesssim \|f\|_{L^{p_0}(\mathbb{F}_q^d,dx)} \quad \text{for all } f : \mathbb{F}_q^d \to \mathbb{C}. \tag{4.3}$$

Since $\max_{x\in V} |f * 1(x)| \leq \|f\|_{L^1(\mathbb{F}_q^d,dx)}$, the inequality (4.2) follows by observing that

$$\begin{aligned}\|f * 1\|_{L^{r_0}(V,d\sigma_v)} &\leq \|f\|_{L^1(\mathbb{F}_q^d,dx)}\|1\|_{L^{r_0}(V,d\sigma_v)} = \|f\|_{L^1(\mathbb{F}_q^d,dx)}\\ &\leq \|f\|_{L^{p_0}(\mathbb{F}_q^d,dx)},\end{aligned}$$

where the last inequality holds, because dx is the normalized counting measure on $\mathbb{F}_q^d$ and $p_0 > 1$. It remains to prove the inequality (4.3). Since we have assumed that $d - \beta - v < 0$, it is not hard to see that the inequality (4.3) follows immediately by interpolating the following two inequalities:

$$\|f * \widehat{K}\|_{L^\infty(V,d\sigma_v)} \lesssim q^{d-s}\|f\|_{L^1(\mathbb{F}_q^d,dx)} \quad \text{for all } f : \mathbb{F}_q^d \to \mathbb{C} \tag{4.4}$$

and

$$\|f * \widehat{K}\|_{L^2(V,d\sigma_v)} \lesssim q^{\frac{d-\beta-v}{2}}\|f\|_{L^2(\mathbb{F}_q^d,dx)} \quad \text{for all } f : \mathbb{F}_q^d \to \mathbb{C}. \tag{4.5}$$

Hence, our task is to prove both (4.4) and (4.5). First let us prove (4.4). For each $x \in V$, it follows that

$$\begin{aligned}|f * \widehat{K}(x)| &\leq \left(\max_{y\in\mathbb{F}_q^d} |\widehat{K}(y)|\right) \frac{1}{q^d} \sum_{y\in\mathbb{F}_q^d} |f(x-y)|\\ &= \left(\max_{y\in\mathbb{F}_q^d} |\widehat{K}(y)|\right) \|f\|_{L^1(\mathbb{F}_q^d,dx)}.\end{aligned}$$

By the definition of K, we see that

$$\max_{y\in\mathbb{F}_q^d} |\widehat{K}(y)| = \max_{y\in\mathbb{F}_q^d} |\sigma_s(y) - \widehat{\delta_{\mathbf{0}}}(y)| = \max_{y\in\mathbb{F}_q^d} \left|\frac{q^d S(y)}{|S|} - 1\right| \leq \frac{q^d}{|S|} \sim q^{d-s},$$

and the inequality (4.4) is obtained. In order to prove (4.5), first notice from the definition of K that $K(m) = (d\sigma_v)^\vee(m)$ for $m \in \mathbb{F}_q^d\setminus\{(0,\ldots,0)\}$

and $K(0,\dots,0)=0$. From this observation and the assumption on the upper bound of $|(d\sigma_v)^\vee(m)|$ for $m\neq(0,\dots,0)$, we see that

$$\max_{m\in\mathbb{F}_q^d}|K(m)|\lesssim q^{-\frac{\beta}{2}}. \tag{4.6}$$

We also need the following restriction estimate.

LEMMA 4.1. *Let $d\sigma_v$ be the normalized surface measure on a variety $V\subset(\mathbb{F}_q^d,dx)$ with $|V|\sim q^v$ for some $0\le v\le d-1$. Then we have*

$$\|\widehat{g}\|_{L^2(V,d\sigma_v)}\lesssim\frac{q^{\frac{d}{2}}}{|V|^{\frac{1}{2}}}\,\|g\|_{L^2(\mathbb{F}_q^d,dm)}\quad\textit{for all functions } g:(\mathbb{F}_q^d,dm)\to\mathbb{C}.$$

Proof. By duality, it is enough to show that the following extension estimate holds:

$$\|(fd\sigma_v)^\vee\|_{L^2(\mathbb{F}_q^d,dm)}\lesssim\frac{q^{\frac{d}{2}}}{|V|^{\frac{1}{2}}}\,\|f\|_{L^2(V,d\sigma_v)}\quad\text{for all functions } f:V\to\mathbb{C}.$$

Recall that we may consider the measure $d\sigma_v$ as a function defined by $\sigma_v(x)=\frac{q^d}{|V|}V(x)$. Therefore, applying the Plancherel theorem yields that

$$\begin{aligned}\|(fd\sigma_v)^\vee\|_{L^2(\mathbb{F}_q^d,dm)}&=\frac{q^d}{|V|}\|(fV)^\vee\|_{L^2(\mathbb{F}_q^d,dm)}=\frac{q^d}{|V|}\|fV\|_{L^2(\mathbb{F}_q^d,dx)}\\&=\frac{q^{\frac{d}{2}}}{|V|^{\frac{1}{2}}}\,\|f\|_{L^2(V,d\sigma_v)}.\end{aligned}$$

The proof is completed. □

Now we are ready to prove the inequality (4.5), which implies the complete proof of Theorem 2.2. Using the property of convolution functions, Lemma 4.1, and (4.6), we conclude that

$$\begin{aligned}\|f*\widehat{K}\|_{L^2(V,d\sigma_v)}&=\|\widehat{f^\vee K}\|_{L^2(V,d\sigma_v)}\lesssim\frac{q^{\frac{d}{2}}}{|V|^{\frac{1}{2}}}\|f^\vee K\|_{L^2(\mathbb{F}_q^d,dm)}\\&\lesssim\frac{q^{\frac{d}{2}}}{|V|^{\frac{1}{2}}}\,q^{-\frac{\beta}{2}}\|f^\vee\|_{L^2(\mathbb{F}_q^d,dm)}=q^{\frac{d-\beta-v}{2}}\|f\|_{L^2(\mathbb{F}_q^d,dx)},\end{aligned}$$

where the Plancherel theorem was also used in the last line. □

Now we restate and give the complete proof of Theorem 2.5.

Theorem 2.5. *Let $S,V\subset\mathbb{F}_q^d$ be algebraic varieties satisfying that $|S|\sim|V|\sim|S\cap V|\sim q^s$ for $\frac{d}{2}<s\le d-1$. In addition, assume that $|(d\sigma_s)^\vee(m)|\lesssim q^{-\frac{s}{2}}$ for all $m\in\mathbb{F}_q^d\setminus\{(0,\dots,0)\}$. Then we have $A_{S,V}(p\to$*

$r) \lesssim 1$ if and only if $(1/p,\, 1/r)$ is contained in the convex hull of points $(0,0), (0,1), (s/d,\, 1)$, and $(s/d,\, (d-s)/d)$.

Proof. ($\Rightarrow$) Suppose that $A_{S,V}(p \to r) \lesssim 1$. From our hypothese, using Corollary 3.3 with $s = v = c$, we see that

$$\frac{d}{p} \leq s \quad \text{and} \quad \frac{d-s}{p} \leq \frac{s}{r}.$$

By simple computation, we conclude that $(1/p,\, 1/r)$ lies on the convex hull of points $(0,0), (0,1), (s/d,\, 1)$, and $(s/d,\, (d-s)/d)$.
($\Leftarrow$) Suppose that $(1/p,\, 1/r)$ is contained in the convex hull of points $(0,0), (0,1), (s/d,\, 1)$, and $(s/d,\, (d-s)/d)$. Note that it is clearly true that $A_{S,V}(\infty \to \infty) \lesssim 1$. We aim to prove that $A_{S,V}(p \to r) \lesssim 1$. Since $\|1\|_{L^k(V,\sigma)} = 1 = \|1\|_{L^k(\mathbb{F}_q^d, dx)}$ for all $1 \leq k \leq \infty$, it follows from Hölder's inequality that

$$A_{S,V}(p_2 \to r) \leq A_{S,V}(p_1 \to r) \quad \text{for} \quad 1 \leq \ p_1 \leq p_2 \leq \infty$$

and

$$A_{S,V}(p \to r_1) \leq A_{S,V}(p \to r_2) \quad \text{for} \quad 1 \leq r_1 \leq r_2 \leq \infty,$$

which reduce our problems to critical endpoint estimates. From this observation and the interpolation theorem, it suffices to prove that $A_{S,V}(d/s \to d/(d-s)) \lesssim 1$. However, this is a direct consequence of Corollary 2.4. Thus, the proof is complete. $\square$

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